MATH 245 F21, Exam 2 Solutions

1. Carefully define the following terms: Proof by Reindexed Induction, Big Omega (Ω)

   To prove $\forall x \in \mathbb{N}, P(x)$ by reindexed induction, we must (base case) prove $P(1)$ is true; and (inductive case) prove $\forall x \in \mathbb{N}$ (with $x \geq 2$), $P(x-1) \rightarrow P(x)$. Given two sequences a_n, b_n, we say that $a_n = \Omega(b_n)$ if there is some $M \in \mathbb{R}$ and some $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $M|a_n| \geq |b_n|$.

2. Carefully state the following theorems: Nonconstructive Existence theorem, well-ordered by $<$

   To prove $\exists x \in D,\ P(x)$ by the Nonconstructive Existence theorem, we prove $\forall x \in D,\ \neg P(x) \equiv F$. Given a set S and an ordering $<$, we say that S is well-ordered by $<$ if every nonempty subset of S has a minimum element, in the $<$ ordering.

3. Prove or disprove: For all $x \in \mathbb{Z}$, there is at most one $y \in \mathbb{Z}$ with $x = 3y^2 + 1$.

   The statement is false. To disprove, we need a counterexample, which consists of a specific x and two different y's (all integers), where x makes the predicate true with *both* y's.

   COUNTEREX 1: Take $x = 4, y_1 = 1, y_2 = -1$. Then $4 = 3(1)^2 + 1 = 3(-1)^2 + 1$.
   COUNTEREX 2: Take $x = 13, y_1 = 2, y_2 = -2$. Then $13 = 3(2)^2 + 1 = 3(-2)^2 + 1$.

4. Prove or disprove: For all $x \in \mathbb{R}$, $\lceil x \rceil \leq \lfloor x \rfloor + 1$.

   The statement is true. Let $x \in \mathbb{R}$ be arbitrary. One of the properties of floor gives $x < \lfloor x \rfloor + 1$, and one of the properties of ceiling gives $\lceil x \rceil - 1 < x$. Combining, we get $\lceil x \rceil - 1 < \lfloor x \rfloor + 1$. Adding 1 to both sides gives $\lceil x \rceil < \lfloor x \rfloor + 2$. Now we apply Thm 1.12 (which we can do since both $\lceil x \rceil$ and $\lfloor x \rfloor + 2$ are integers), to get $\lceil x \rceil \leq (\lfloor x \rfloor + 2) - 1 = \lfloor x \rfloor + 1$.

5. Use mathematical induction to prove: For all $n \in \mathbb{N}$, $n! \geq n$.

   Base case: $n = 1$. We have $1! = 1$, so $n! = 1! = 1 \geq 1 = n$.
   Inductive case: Let $n \in \mathbb{N}$ be arbitrary, and suppose that $n! \geq n$. We multiply both sides by $(n+1)$, getting $(n+1) \cdot n! \geq n(n+1)$. Now, $(n+1) \cdot n! = (n+1)!$, and $n(n+1) \geq 1(n+1) = n+1$ (since $n \geq 1$). We conclude $(n+1)! \geq n+1$.

6. Solve the recurrence with initial conditions $a_0 = 1, a_1 = 2$, and recurrence relation $a_n = -4a_{n-1} - 4a_{n-2}$ ($n \geq 2$).

   Our characteristic polynomial is $r^2 + 4r + 4 = (r+2)^2$. Hence there is a double root of -2, and our general solution is $a_n = A(-2)^n + Bn(-2)^n$. We now apply the initial conditions to get $1 = a_0 = A(-2)^0 + B \cdot 0 \cdot (-2)^0 = A$, and $2 = a_1 = A(-2)^1 + B \cdot 1 \cdot (-2)^1 = -2A - 2B$. This has solution $\{A = 1, B = -2\}$, so the specific solution is $a_n = 1 \cdot (-2)^n - 2n(-2)^n = (-2)^n + n(-2)^{n+1}$.

7. Let $b_n = 100 + 4n$. Prove or disprove that $b_n = O(n)$.

The statement is true.

PROOF 1: Let $M = 5, n_0 = 100$. Let $n \geq n_0 = 100$ be arbitrary. Add $4n$ to both sides, getting $4n + n \geq 4n + 100$. Hence, $|b_n| = |100 + 4n| = 100 + 4n \leq 4n + n = 5n = M|n|$.

PROOF 2: Let $M = 104, n_0 = 1$. Let $n \geq n_0 = 1$ be arbitrary. Multiply by 100 on both sides, getting $100n \geq 100$. Add $4n$ to both sides, getting $100n + 4n \geq 100 + 4n$. Hence, $|b_n| = |100 + 4n| = 100 + 4n \leq 100n + 4n = 104n = M|n|$.

8. Prove that, for all $x \in \mathbb{R}$, $|x| + |x - 2| \geq 2$.

Let $x \in \mathbb{R}$ be arbitrary. We now have three cases, depending on whether $x \leq 0$, or $0 < x \leq 2$, or $x > 2$. These are the three cases you must use to solve the problem correctly (apart from the boundary points $x = 0$,$x = 2$, which you can change around).

Case $x \leq 0$: $|x| + |x - 2| = -(x) - (x - 2) = -2x + 2$. Since $x \leq 0$, we multiply by -2 to get $-2x \geq 0$. Adding 2, we get $-2x + 2 \geq 0 + 2 = 2$. Hence $|x| + |x - 2| = -2x + 2 \geq 2$.

Case $0 < x \leq 2$: $|x| + |x - 2| = +(x) - (x - 2) = 2$. (and $2 \geq 2$).

Case $x > 2$: $|x| + |x - 2| = +(x) + (x - 2) = 2x - 2$. Since $x > 2$, we multiply by 2 to get $2x > 4$. Adding -2 to both sides, we get $2x - 2 > 4 - 2 = 2$. Hence $|x| + |x - 2| = 2x - 2 > 2$, and thus $|x| + |x - 2| \geq 2$.
(note that $(a > b) \vdash (a > b) \vee (a = b)$ by addition, and $(a > b) \vee (a = b) \equiv (a \geq b)$).

9. Prove: For all $n \in \mathbb{Z}$ with $n \geq 2$, that $F_n \geq 2F_{n-2}$. (Here F_n denotes the Fibonacci numbers) NOTE: Unfortunately, the exam as given had a typo, where $F_n \geq 2F_{n-2}$ was incorrectly given as $F_n > 2F_{n-2}$. This ruins the second base case (everything else is fine). Any student tripped up by this typo had their grade compensated.

Here we need strong induction and two base cases.

Base case $n = 2$: $F_2 = 1, F_0 = 0$, and hence $F_2 = 1 \geq 0 = 2F_0$.
Base case $n = 3$: $F_3 = 2, F_1 = 1$, and hence $F_3 = 2 \geq 2 = 2F_1$.

Inductive case: Let $n \in \mathbb{Z}$ with $n \geq 4$, and suppose that the predicate is true for all smaller n (that are at least 2). In particular, it is true for $n - 1$ and $n - 2$. Hence $F_{n-1} \geq 2F_{n-3}$ and $F_{n-2} \geq 2F_{n-4}$. We add these inequalities, getting $F_{n-1} + F_{n-2} \geq 2F_{n-3} + 2F_{n-4} = 2(F_{n-3} + F_{n-4})$. Now, we use the defining recurrence relation of Fibonacci numbers twice: $F_{n-1} + F_{n-2} = F_n$, and $F_{n-3} + F_{n-4} = F_{n-2}$. Substituting, we get $F_n \geq 2F_{n-2}$.

10. Use maximum element induction to prove that $\forall x \in \mathbb{N}, \exists n \in \mathbb{N}_0,\ 2^n \leq x < 2^{n+1}$.

For any $x \in \mathbb{N}$, set $S = \{n \in \mathbb{N}_0 : 2^n \leq x\}$. Note that S is nonempty, because $0 \in S$ (since $2^0 = 1 \leq x$). The algebraically hard part of this problem is to prove that S has an upper bound(*). Once we do this, then maximum element induction gives us a maximum element n of S, such that n satisfies the predicate but $n + 1$ does not, i.e. $2^n \leq x$ and $2^{n+1} \not\leq x$. Combining, we get $2^n \leq x < 2^{n+1}$.

(*) To show that S has an upper bound: If $2^n \leq x$, then $n \leq \log_2 x$ (since $\log_2$ is an increasing function, we can apply it to both sides of the inequality). Hence $\log_2 x$ is an upper bound for S, because every $n \in S$ must satisfy $2^n \leq x$, and must therefore also satisfy $n \leq \log_2 x$.